B.A. /B.Sc. Part-III (Honours) Examination, 2020 (1+1+1)
Subject: Mathematics
Paper: V

Time: 2 Hours Full Marks: 50

The figures in the margin indicate full marks.
Candidates are required to write their answers in their own words as far as practicable.
[Notation and Symbols have their usual meaning]

1. Answer any ***four*** questions: $4 \times 5 = 20$

(a) Give the definition of Riemann integral of a bounded function f over a closed interval $[a,b]$. If f is a monotone increasing function on $[a,b]$, then show that $\int_a^b f dx$ exists in Riemann sense. 1+4

(b) Test for the uniform convergence of the sequence of functions $\{f_n\}$ defined by $f_n(x) = nx(1-x)^n, x \in [0,1]$ 5

(c) State Abel's theorem for the convergence of an improper integral. Examine the absolute convergence of $\int_0^1 \frac{1}{\sqrt{x}} \sin\frac{1}{x} dx$ 1+4

(d) Find the extreme values of the function $f(x, y) = x^3 - y^3 + 3x^2 + 3y^2 - 9x, (x, y \in \mathbb{R})$ 5

(e) Suppose $f(z) = \begin{cases} \frac{(\bar{z})^2}{z}, & z \neq 0 \\ 0, & z = 0 \end{cases}$ be a complex valued function of a complex variable z. Examine if Cauchy- Riemann equations are satisfied at $z = 0$. 5

(f) Define closure and interior of a set A. Show that $\bar{A} = X \setminus Int(X \setminus A)$, where $\bar{A}$ and $Int(A)$ are the closure and interior of a set A respectively. 2+3

2. Answer any ***three*** questions: $3 \times 10 = 30$

(a) (i) Prove that every metric space is first countable.

(ii) Prove that a metric space is second countable if and only if it is separable. 4 +6

(b) (i) State and prove Cauchy-Hadamard theorem on the radius of convergence of a power series in a complex plane.

(ii) Find the bilinear transformation which maps the points $z_1 = 2, z_2 = i \; and \; z_3 = -2$ into the points $w_1 = 1, w_2 = i \;\; and \;\; w_3 = -1$. 6+4

(c) (i) State Dirichlet's conditions concerning convergence of Fourier series of a function.

(ii) If f be a periodic function of period 2π such that $f(x) = x^2, 0 \le x < 2\pi$. find the Fourier series of f and examine the convergence of the series in $(0, 2\pi)$ and at the points $x = 0,\; 2\pi.$ 2+(5+3)

(d) (i) If $f(x, y)$ is differentiable at the point $(a,b) \in \mathbb{R} \times \mathbb{R}$ then show that $f(x, y)$ is continuous at (a,b). Show that the function f where

$$\begin{cases} f(x,y) = x\sin\dfrac{1}{x} + y\sin\dfrac{1}{y}, & (x,y) \ne (0,0) \\ x\sin\dfrac{1}{x} & , x \ne 0, y = 0 \\ y\sin\dfrac{1}{y} & , y \ne 0, x = 0 \\ 0 & , (x,y) = (0,0) \end{cases}$$

is continuous but not differentiable at the origin.

(ii) State and prove Euler's theorem on homogeneous functions for a function of two variables. (2+3)+ (1+4)

(e) (i) Evaluate : $\iint_D x^2 y^2 dxdy$, where $D = \{(x, y) : x \ge 0, y \ge 0; x^2 + y^2 \le 1\}$, x, y are reals.

(ii) Prove the relation $\beta(m,n) = \dfrac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}, m > 0, n > 0.$ 3+7